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Zero-branes on a compact orbifold

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Abstract: The non-commutative algebra which defines the theory of zero-branes on T^4/Z_2 allows a unified description of moduli spaces associated with zero-branes, two-branes and four-branes on the orbifold space. Bundles on a dual space $\hat{T}^4/Z_2$ play an important role in this description. We discuss these moduli spaces in the context of dualities of K3 compactifications, and in terms of properties of instantons on T^4. Zero-branes on the degenerate limits of the compact orbifold lead to fixed points with six-dimensional scale but not conformal invariance. We identify some of these in terms of the ADS dual of the $(0,2)$ theory at large N, giving evidence for an interesting picture of "where the branes live" in ADS.

Keywords: Superstrings and Heterotic Strings, M-Theory, D-branes, String Duality.

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Contents

1. Introduction

D-branes with worldvolume transverse to ALE orbifolds, R^4/G, in type II string theories have been studied in [1]-[9], while 0-branes on tori R^p/Z^p have been studied in [10, 11]. The theory is derived by finding a representation of the discrete group G or the translation group Z^p. The representation space also admits an action of operators X corresponding to zero-brane positions. These operators satisfy constraints determined by the group G or Z^p. The case of compact orbifolds, R^4/Γ, where Γ is the semidirect product of G and Z^p, studied in [12] has the novel feature that the translation operators do not commute with the discrete group. In this paper we study, in detail, the case of T^4/Z_2.

In section 2, we review the theory of zero-branes on a torus T^p. In particular, we recall how to derive a $p+1$ dimensional theory from a solution of the translation constraints, and how these theories produce the moduli space $(T^p)^N/S_N$ for N zero-branes. This field theory can be viewed as a world-volume theory of Dp branes, and the relation between 0 and p branes is the one expected from T-duality.

In section 3, we review some facts about dualities of $K3$ compactifications of Type IIA string theory, as summarized in [13], focusing on the perturbative orbifold limit.

In section 4, we describe the theory of N pure zero-branes on T^4/Z_2 It is obtained by finding X operators acting in the regular representation of the constraint algebra, which we call Rep. I. We check that the moduli space of multi-zero-branes on the orbifold is correctly reproduced. We also check that the appropriate enhancement of the Coulomb branch occurs when a zero-brane hits any of the sixteen fixed points. We give a description of the resulting theory as a $U(2N)$ gauge theory on $\hat{T}^4/Z_2$, where $\hat{T}^4$ is the T-dual of T^4 with inverse volume. The spatial Z_2 reflection is combined with a non-trivial gauge transformation, resulting in a non-trivial gauge bundle on the orbifold.

In section 5, we study another representation of the algebra of constraints, which we call Rep. II. We interpret it in terms of N two-branes at a fixed point on T^4/Z_2, and check that the moduli spaces are consistent with this interpretation. We give a description in terms of $U(N)$ gauge theory on $\hat{T}^4/Z_2$, where Z_2 is embedded trivially in the gauge group.

In section 6, we study a class of other representations of the constraint algebra which are characterized algebraically by the different representatives of the generator of the Z_2 group. The physical interpretation is most simply given in terms of states with wrapped membrane and anti-membrane charge at different fixed points of T^4/Z_2. We check that the Higgs and Coulomb branches have the form expected from this physical interpretation.

In section 7, we compare the relation between the theories on T^4/Z_2 and $\hat{T}^4/Z_2$ to relations we might expect from dualities of $K3$ compactifications. Some facts about these dualities are collected in Appendix A. We write down the relevant element of $O(\Gamma_{4,20})$. A lot of its intricate structure is deduced from properties of the gauge theory description of branes on the two spaces. There is a relation to the point-like instantons at fixed points of [14, 15].

In section 8, we study relations between the moduli spaces of vacua of the theories we construct and moduli spaces of instantons on T^4, invariant under certain Z_2 symmetry groups.

In section 9, we describe the construction of six dimensional fixed points by taking certain limits of the parameters of the orbifold compactification. We give the ADS interpretation of these fixed points, following the correspondence between ADS_7 and the large N $(0,2)$ theory proposed by Maldacena [16], and elaborated on by [17, 18].

2. Review of related quotients

The theory of N zero-branes in flat space is described by the action:

$$L = \frac{1}{2g}\int dt \,\mathrm{tr}\left[D_t X^i D_t X^i + 2\theta^T D_t\theta - \frac{1}{2}[X^i, X^j]^2 - 2\theta^T\gamma_i[\theta, X^i]\right], \tag{2.1}$$

where $D_t = \partial_t - iA_0$. The general procedure for describing the zero-branes on a quotient of R^p is to find a representation of the quotienting group acting on the fields X. Let us review this in two cases.

2.1. $T^4 = R^4/Z^4$

The theory of N zero-branes on T^4 is defined by starting with the quantum mechanics of zero-branes and imposing constraints.

$$\begin{aligned} U_a X^b U_a^{-1} &= X^b + 2\pi R_a \delta^{ab}\,, \\ U_a X^i U_a^{-1} &= X^i. \end{aligned} \tag{2.2}$$

Here the U_a represent the 4 discrete translations generating the quotienting group Z^4, and thus commute with each other. The X^a are related to positions of the zero-branes on the torus and X^i are related to transverse positions.

The constraints (2.2) can be solved by writing

$$\begin{aligned} X^a &= i\partial^a + A^a(x) \\ X^i &= X^i(x) \\ U_a &= e^{2\pi i R_a x_a}\,, \end{aligned} \tag{2.3}$$

where x^a are periodic variables

$$x^a \sim x^a + \frac{1}{R_a}, \tag{2.4}$$

i.e. they are coordinates on the dual torus $\hat{T}^4$. In the simplest case the gauge fields, and adjoint matter X^i obey periodic boundary conditions. We have then, a *local theory* with $U(N)$ *gauge symmetry* on the T-dual torus $\hat{T}^4$. This theory has a moduli space of $(T^4 \times R^5)^N/S_N$ as expected for zero branes on the space $T^4 \times R^5$.

This $4+1$ dimensional Yang Mills theory can be viewed as the world-volume theory of N four-branes on $\hat{T}^4$. The correspondence between the zero-branes on T^4 and the 4-branes on $\hat{T}^4$ is just T-duality on all 4 circles of the torus, an element of the T-duality group $O(4,4,Z)$ of the torus, as reviewed for example in [19].

More generally one considers non-trivial bundles characterized by non-zero Chern classes $\mathrm{tr}[X_a, X_b]$ and $\mathrm{tr}\epsilon^{abcd}X_aX_bX_cX_d$, which correspond to systems of zero-branes on T^4, together with extra 2 and 4-branes respectively [20, 21]. The appearance of moduli spaces $S^N(T^4)$ in these more general systems was studied in [22]. One can also consider a generalization where the elements corresponding to translations are represented by operators which only commute up to a phase. These are related to backgrounds with B-fields [23], and lead to non-commutative gauge theories.

2.2. R^4/Z_2

The quotient by a discrete Z_2 is described by:

$$\begin{aligned} \Omega X^a \Omega &= -X^a \\ \Omega X^i \Omega &= X^i\,, \end{aligned} \tag{2.5}$$

where $\Omega^2 = 1$. A number of interesting properties of the moduli spaces corresponding to the coordinates transverse ("Coulomb branches") as well as to coordinates parallel ("Higgs branches") to the R^4/Z_2 ALE space were elucidated in [1, 24, 25, 26].

For a single zero-brane, there is a Higgs branch R^4/Z_2 corresponding to the zero-brane moving on the ALE. For a generic point on the Higgs branch, the Coulomb branch is R^5. When the zero brane hits the fixed point, the Coulomb branch is enhanced to $(R^5)^2$. In order for the ALE space to be a perturbative string vacuum, there must be 1/2 a unit of B-field [27] on the collapsed two-cycle at the singularity. The enhanced Coulomb branch can then be interpreted in terms of a pair of states which carry opposite membrane charge, wrapped on the fixed point, and having a net zero-brane charge, because of the coupling

$$\int C \wedge (F - B) \tag{2.6}$$

on the world-volume of the two-brane. We will be more precise about the contributions to the zero-brane charge from F and B in section four. C is the one-form potential coupling to a zero-brane.

3. Review of duality on T^4/Z_2

In this paper we will construct the theory of zero-branes on T^4/Z_2. By analogy with the construction on T^4 we might expect the description to be a gauge theory on some dual compact orbifold, exchanging zero-brane charge for four-brane charge and inverting the volume of the orbifold. In this section we will briefly review what kind of duality transformation we might expect.

There are two obvious candidates. A first possibility is to make a T-duality on the covering torus T^4 to a dual torus $\hat{T}^4$ (with inverse volume) and then project again to $\hat{T}^4/Z_2$. Alternatively, one can view T^4/Z_2 as an orbifold limit of a smooth K3 manifold, with sixteen A_1 singularities. For $U(N)$ gauge theories with instanton number k on K3, it is known that there is a generalization of Nahm duality for T^4 [28], the Fourier-Mukai (FM) transform [29, 1, 30, 31], which exchanges N with $k-N$.

It is important to note that the two candidate dualities are not the same. In the case of T-duality inherited from the double cover T^4, for one zero-brane on T^4/Z_2 we need two zero-branes on the T^4. Under duality this should turn into two four-branes on $\hat{T}^4$, described by a $U(2)$ gauge theory. Modding out by Z_2 the theory remains $U(2)$. Thus we see that a single zero-brane on T^4/Z_2 should map to two four-branes on $\hat{T}^4/Z_2$. One notes further that the dual torus $\hat{T}^4$ should have the inverse volume of the original T^4. Thus if V is the volume of the original T^4/Z_2 then the dual space $\hat{T}^4/Z_2$ must have volume $1/4V$ rather than simply $1/V$.

As for the ALE singularity, the geometrical orbifold is not a perturbative string background. The perturbative background has one-half a unit of B-field on each collapsed two-cycle [27]. Since we construct the zero-brane theory from perturbative open strings, we are really concerned with duality of the T^4/Z_2 orbifold with B-field on the collapsed cycles.

The Mukai charge (Q_4, Q_2, Q_0) of a system of branes characterizes Q_4 four-branes, described by $U(Q_4)$ gauge theory, with magnetic flux Q_2 giving the number of two-branes, and with zero-brane charge Q_0. Q_0 includes a contribution from the instanton number, measuring the number of physical zero-branes, together with $-Q_4$ induced from the curvature of the $K3$ [32, 33, 34]. Under FM duality, Q_4 and Q_0 are exchanged. Thus, in contrast to the T-duality inherited from T^4, a single zero-brane transforms to a system of *one* four-brane with instanton number one.

To understand how the volume of the K3 transforms under Fourier-Mukai duality, one identifies the duality with a particular element of the duality group of IIA string theory on a K3. The structure of the string moduli space has been described by Aspinwall [13] and is summarized in the Appendix. The relevant point here is that the volume of the K3 is encoded in a normalized vector B' in the total cohomology $H^*(K3,R) \cong R^{4,20}$. The integer cohomology forms a even self-dual lattice $H^*(K3,Z) \cong \Gamma_{4,20} \subset R^{4,20}$. Points in the lattice represent the Mukai charge of a system of four-branes, membranes and zero-branes wrapped on the cycles defined by the point in $\Gamma_{4,20}$. There is a group of discrete rotation symmetries which leave the lattice invariant and define the same physical system. These are the dualities of the theory. If we write ω as the single element generating $H^0(K3,Z)$ and ω^* as the dual element generating $H^4(K3,Z)$, then, if there is no B-field on the K3, the vector defining the volume is given by

$$B' = \omega^* + \alpha\omega \tag{3.1}$$

(this vector is always normalized so that the coefficient of ω^* is one), and α is the volume of the K3. Similarly an element of the lattice which describes only a sum of zero-brane and four-brane Mukai charge is

$$p = Q_4\omega^* + Q_0\omega\,. \tag{3.2}$$

One of the dualities which leaves the $\Gamma_{4,20}$ lattice invariant is the exchange of ω and ω^*. Writing $\omega \to \bar{\omega} = \omega^*$ and $\omega^* \to \bar{\omega}^* = \omega$ we find

$$\begin{aligned} p &= Q_0\bar{\omega}^* + Q_4\bar{\omega} \\ B' &= \bar{\omega}^* + \bar{\omega}/\alpha\,, \end{aligned} \tag{3.3}$$

where we have rescaled B' to get the correct normalization with respect to $\bar{\omega}^*$. We see that this duality has realized the Fourier-Mukai transformation, exchanging the zero- and four-brane Mukai charges. Since $\alpha \to 1/\alpha$, in contrast to the T-duality from the torus where $V \to 1/4V$, under Fourier-Mukai duality we have $V \to 1/V$.

For simplicity we did not include the B-field in the above discussion. This is done in the Appendix. We also note that one might expect that the duality inherited from the T-duality of T^4 is also realized as a symmetry of the $\Gamma_{4,20}$ lattice. We will return to these questions later.

4. The regular representation (Rep. I)

4.1. Algebra, solutions and non-commutativity

The space T^4/Z_2 can be thought of as the plane R^4 modded out by the group Γ generated by translations U_a (where a runs from 1 to 4) and an inversion Ω. These generators satisfy the conditions

$$\Omega U_a \Omega = U_a^{-1}, \qquad \Omega^2 = 1, \tag{4.1}$$

and otherwise commute.

A general element of the quotienting group is given by

$$U_g = \prod_a U_a^{n_a}\Omega^p, \qquad n_a \in Z, \quad p \in \{0,1\}. \tag{4.2}$$

This is naturally extended to the group algebra by considering combinations with complex coefficients.

To write an action for D0-branes on T^4/Z_2 we need to find a set of operators such that

$$\begin{aligned} U_a X^b U_a^{-1} &= X^b + 2\pi R_a \delta_a{}^b \\ \Omega X^a \Omega &= -X^a \\ U_a X^i U_a^{-1} &= X^i \\ \Omega X^i \Omega &= X^i \,. \end{aligned} \tag{4.3}$$

A procedure to find the solution of these conditions in the regular representation, for arbitrary constraint algebra constructed from translations and discrete subgroups of rotations, is given in [12]. One introduces a set of dual operators $\tilde{U}_g$ and the solution for X^a and X^i becomes a general function in the $\tilde{U}_g$. In our case, one finds that the $\tilde{U}_g$ satisfy the same algebra as the original U_g.

4.2. Explicit representation

Here we will give an explicit solution of (4.3). Let us take

$$\Omega = \begin{pmatrix} * & 0 \\ 0 & -* \end{pmatrix} , \qquad U_a = e^{2\pi i R_a x^a} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{4.4}$$

acting on the space of pairs of functions on $\hat{T}^4$. The operator $*$ acts on functions as $*f(x) = f(-x)*$. We then have

$$X^a = i\partial^a + \begin{pmatrix} A^a & B^a \\ B^{\dagger a} & \hat{A}^a \end{pmatrix} \qquad X^i = \begin{pmatrix} X^i & Y^i \\ Y^{\dagger i} & \hat{X}^i \end{pmatrix} \tag{4.5}$$

Where A^a, $\hat{A}^a$ and Y^i are odd functions under $x^a \to -x^a$ while B^a, X^i and $\hat{X}^i$ are even. Also A^a, $\hat{A}^a$, X^i and $\hat{X}^i$ are all real. In general, to describe N D0-branes all these functions are replaced by $N \times N$ matrices.

This corresponds to the regular representation discussed in [12]. This representation is a Hilbert space with a distinct basis state for each element of the group Γ, i.e.

$$\big|U_g\big\rangle = U_g \big|0\big\rangle \tag{4.6}$$

for each $U_g \in \Gamma$.

In the representation above, the vacuum $\big|0\big\rangle$ is identified with the pair of functions

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

If we choose the action

$$\Omega \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} ,$$

states in the Hilbert space can be mapped to pairs of functions.

The solution can be expressed in the language of [12]:

$$\begin{aligned} X^a &= A^a(\tilde{U}^a,\tilde{\Omega})\tilde{R} + \tilde{d}^a \\ X^i &= X^i(\tilde{U}^a,\tilde{\Omega})\,, \end{aligned} \tag{4.7}$$

where $\tilde{\Omega}$ and $\tilde{U}$ generate the commutant of the algebra of U and Ω in this Hilbert space and form an isomorphic algebra. (In the more general representations to be discussed later, the commutant is not an algebra generated by such a set of generators). The operators $\tilde{R}$ and $\tilde{d}$ satisfy $\tilde{R}f(\tilde{U}^a,\Omega) = f(\tilde{U}^a,-\Omega)\tilde{R}$, $\tilde{R}^2 = 1$ and $\tilde{d}^a\tilde{U}^b = \tilde{U}^b\tilde{d}^a + \tilde{U}^b\tilde{R}\delta^{ab}R_a$. These can be expressed as

$$\begin{aligned} \tilde{\Omega} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad & \tilde{U^a} &= \begin{pmatrix} \cos(2\pi Rx^a) & i\sin(2\pi Rx^a)\,, \\ i\sin(2\pi Rx^a) & \cos(2\pi Rx^a) \end{pmatrix}, \\ \tilde{R} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad & \tilde{d^a} &= \begin{pmatrix} i\partial^a & 0 \\ 0 & i\partial^a \end{pmatrix}. \end{aligned} \tag{4.8}$$

Including A_0, which has a similar form to X^i, we obtain the action on the dual torus $\hat{T}^4$. We find, in a $U(2)$ invariant form

$$S = \int_{\hat{T}^4} \left\{ -\frac{1}{4g}\mathrm{tr}F_{ab}F^{ab} - \frac{1}{2g}D_aX^iD^aX_i \right\}, \tag{4.9}$$

where

$$D_aX^i = \partial_aX^i - i[A_a,X^i] \tag{4.10}$$

and F is the usual field strength for A_a considered as a $U(2)$ field.

The action is not in fact invariant under arbitrary $U(2)$ gauge transformations on $\hat{T}^4$. Only those gauge transformations which commute with the constraint (i.e. with Ω) are preserved. This means that a general infinitesimal gauge parameter Λ is given by

$$\Lambda = \begin{pmatrix} \lambda & \mu \\ \mu^\dagger & \hat{\lambda} \end{pmatrix}, \tag{4.11}$$

where λ and $\hat{\lambda}$ are even functions while μ is odd. The condition satisfied by the gauge transformations is *not local* on $\hat{T}^4$.

Since all the fields have a definite parity under $x^a \to -x^a$, the action can be further reduced to one on $\hat{T}^4/Z_2$. We find

$$\begin{aligned} S = \int_{\hat{T}^4/Z_2} \Big\{ &-\frac{1}{2g}\mathrm{tr}\left(F - iB\wedge B^\dagger\right)^2 - \frac{1}{2g}\mathrm{tr}\left(\hat{F} - iB^\dagger\wedge B\right)^2 - \frac{1}{g}\mathrm{tr}(DB)(DB^\dagger) \\ &-\frac{1}{g}\mathrm{tr}\left(DX - iBY^\dagger + iYB^\dagger\right)^2 - \frac{1}{g}\mathrm{tr}\left(D\hat{X} - iB^\dagger Y + iY^\dagger B\right)^2 \\ &-\frac{2}{g}\mathrm{tr}\left(DY - iB\hat{X} + iXB\right)\left(DY^\dagger - iB^\dagger X + i\hat{X}B^\dagger\right)\Big\}, \end{aligned} \tag{4.12}$$

where we have the covariant derivatives

$$\begin{aligned}
DB &= dB - iA \wedge B - iB \wedge \hat{A} \\
DX^i &= dX^i - i[A, X^i] \\
D\hat{X}^i &= dX^i - i[\hat{A}, \hat{X}^i] \\
DY^i &= dY^i - iAY^i + iY^i\hat{A} \,.
\end{aligned} \tag{4.13}$$

We see that A and $\hat{A}$ are gauge fields (in general $U(N) \times U(N)$). For the scalar matter fields, X^i and $\hat{X}^i$ are in the adjoint of the left and right $U(N)$ respectively, while Y^i is in the fundamental $(\mathbf{N}, \bar{\mathbf{N}})$ of each group. Similarly B_a is a one-form charged in the fundamental of each group. This structure is similar to the $U(N) \times U(N)$ gauge symmetry for zero-branes on R^4/Z_2.

The theory on $\hat{T}^4/Z_2$ is a $U(2N)$ gauge theory, with boundary conditions which leave a manifest $U(N) \times U(N)$ gauge symmetry. The massless vector fields are $U(N) \times U(N)$ gauge fields. The B^a fields appear as massive spin-one matter. That we can write a local theory on $\hat{T}^4/Z_2$ is related to the fact that the fields can be viewed as sections of a non-trivial bundle on that space. The non-triviality comes from the gauge transformation by $\sigma_3 \otimes 1$ which accompanies the transition from patches related by reflection in the fixed points, where σ_i are the Pauli matrices. Strictly speaking the bundle has a singularity at the fixed points, in the sense that the patching functions fail to obey the condition $U_{\alpha\beta}U_{\beta\gamma}U_{\gamma\delta} = 1$ at the fixed points.

The singularity of the bundle structure in the neighbourhood of a fixed point can be characterized by a Wilson loop on a non-trivial path in the space with the point removed. This has non-trivial π_1 since it is retractable to S^3/Z_2 which has $\pi_1(S^3/Z_2) = Z_2$. Along the lines of [15] we expect that this should be related to the existence of two-brane charge at the collapsing 2-cycle of the blown-up space. In the blown-up space, this would be realized as $\int \mathrm{tr} F$ on the 2-cycle. In this sense the breaking from $U(2N)$ to $U(N) \times U(N)$ may be understood as a result of having a non-trivial background at the fixed point. Typically there would also be a contribution to the instanton number at the fixed point from this background.

4.3. Moduli space for one zero-brane

Since the theory we described above has some rather peculiar properties, it is useful to demonstrate that it is indeed describing zero-branes on T^4/Z_2. We show here that we do get a Higgs branch moduli space of T^4/Z_2, together with the enhanced Coulomb branches whenever the zero-brane hits any fixed point.

Since we are looking for zero-energy configurations, it will suffice to consider configurations of zero field-strength $F = 0$. Recall how these give rise to the expected moduli space if we just had the $U(2)$ theory of two zero-branes on T^4. In that case, the moduli space is described by constant gauge fields A_a and scalars X^i. We can simultaneously diagonalize the $U(2)$ matrices, and allowing for the residual Z_2 symmetry, we have a moduli space of $(T^4 \times R^5)^2/Z_2$. This is the configuration space of two identical zero branes on the space $T^4 \times R^5$.

In this case the reduced gauge symmetry means that we cannot diagonalize the $U(2)$ matrices. Consider constant gauge connection. Since A and $\hat{A}$ are odd functions this means

$$A^a = \begin{pmatrix} 0 & B^a \\ \bar{B}^a & 0 \end{pmatrix} = B_1^a \sigma_1 + B_2^a \sigma_2 \,. \tag{4.14}$$

Similarly the allowed constant gauge transformations are

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \hat{\lambda} \end{pmatrix} = \lambda_0 1 + \lambda_3 \sigma_3 \tag{4.15}$$

These are not enough to diagonalize A^a, at best we can rotate all the field into, say σ_1 so that

$$A^a = B_1^a \sigma_1 \tag{4.16}$$

and there is still a residual gauge symmetry σ_3 which identifies B_1^a with $-B_1^a$.

Now we can also identify B_1^a with $B_1^a + 2\pi R^a$ (a translation on the original torus) since these configurations are related by gauge transformations with

$$g = \exp(2\pi i R_a x^a \sigma_1) \tag{4.17}$$

(Note that the exponent is an odd function of x^a and so this is an allowed gauge transformation.) This, together with the residual gauge symmetry σ_3, leads to a Higgs branch moduli of T^4/Z_2.

Now we turn to the X^i sector. We need a solution where

$$\partial_a X^i - i[A_a, X^i] = 0, \qquad [X^i, X^j] = 0 \tag{4.18}$$

where $A^a = B_1^a \sigma_1$. We can find solutions by the following procedure. For $A_a = 0$ we clearly have the solution

$$X^i = X_0^i 1 + X_3^i \sigma_3 \tag{4.19}$$

where X_0^i and X_3^i are constant (which is allowed given the odd/even properties X^i). In such a case the transverse moduli space is $R^5 \times R^5$. We can grow a solution with $A_a \neq 0$ by making a gauge transformation on the covering R^4 of the form (which has the correct odd/even properties)

$$g = \exp(i B_1^a x_a \sigma_1) \tag{4.20}$$

giving $A^a = B_1^a \sigma_1$. In general such a gauge transformation on R^4 is not a valid gauge transformation on $\hat{T}^4$ since it fails to be single valued on $\hat{T}^4$. However, we can use it to generate solutions provided the resulting fields are properly single valued.

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To check single-valuedness, we write the transformed X^i :

$$X^i = X_0^i 1 + X_3^i \exp(2iB_1^a x_a \sigma_1)\sigma_3 \tag{4.21}$$

For general B_1^a this is only single valued if $X_3^i = 0$. Thus for generic points in T^4/Z_2 the transverse moduli space is R^5. The fixed points of the orbifold correspond to $B_1^a = \pi k^a$, where k^a is a vector with components 0 or R_a. We then see that at the fixed points, there is no constraint on $X_3^i = 0$ and the moduli space becomes $R^5 \times R^5$.

The moduli space may be viewed as a fibration over the Higgs branch of T^4/Z_2. The fibre at a generic point is R^5. However at the orbifold points, the fibre becomes $R^5 \times R^5$. Note that this is $R^5 \times R^5$ rather than $(R^5 \times R^5)/Z_2$, which is consistent with the interpretation that the zero-brane decays into two distinct objects, free to move only transverse to the fixed point. Inspection of the mass formula in the appendix, shows that the four states with Mukai vectors γ_{16}, $-\gamma_{16}$, $\gamma_{16} - \omega$ and $-\gamma_{16} + \omega$, are degenerate and are the lightest states carrying membrane charge on the collapsed cycle γ_{16}. Each state has half the mass of a single zero-brane. Taking into account the B field, these states carry total charge $(q_2^{(16)}, q_0)$ of $(1, \frac{1}{2})$, $(-1, -\frac{1}{2})$, $(1, -\frac{1}{2})$ and $(-1, \frac{1}{2})$ respectively. Thus a zero-brane can decay into the pair $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$ with the same charge and mass. In the rest of the paper, we will refer to the state $(1, \frac{1}{2})$ as "membrane" and the state $(-1, \frac{1}{2})$ as "antimembrane".[1]

4.4. $S^N(T^4/Z_2)$ for N zero-branes

The discussion of the Higgs branch where $X^a \neq 0$ and $X^i = 0$ generalizes in a simple way. As before we can get the expected moduli space by restricting attention to zero field-strength configurations (We expect that the condition of zero-energy will force the X^a to be space-time independent up to gauge transformations, along the lines of [35]).

Any constant matrix X^a of the form

$$\begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}$$

can be put in a standard form

$$\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \tag{4.22}$$

where D is diagonal and real, using gauge transformations of the form

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

[1] In terms of the Mukai vectors, we note that the $(-1, -\frac{1}{2})$ state is more strictly the pure anti-membrane state. However since in what follows we will only be interested in the state $(-1, \frac{1}{2})$, which actually has Mukai vector $-\gamma_{16} + \omega$, for simplicity, we will refer to this latter state as the "antimembrane".

This follows from using the Polar Decomposition theorem [36]. Then any other matrix of the form

$$\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix},$$

which commutes with X^a can simultaneously be put in the form (4.22). Permutation matrices of the form

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

and the matrix diag$(1, -1)$ lead to identifications which leave a moduli space with a Higgs branch $(T^4/Z_2)^N/S_N$. This is as expected for a system of N zero-branes on T^4/Z_2.

5. Rep. II — Interpretations on T^4/Z_2 and $\hat{T}^4/Z_2$

In this section we study Rep. II given by the following solution of (4.3)

$$\begin{aligned} U_a &= e^{2\pi i R_a x_a} \\ \Omega &= * \\ X_a &= iD_a(x) \\ X_i &= X_i(x)\,, \end{aligned} \tag{5.1}$$

which is even simpler than Rep. I. The Ω constraint requires that the gauge field $A_a(x)$ is odd, the scalar $X_i(x)$ is even, and the gauge parameter is even. The gauge symmetry is $U(1)$ in the simplest case. One generalizes to $U(N)$ by replacing $\Omega = *$ by $\Omega = *1$, where 1 is the identity matrix in $U(N)$, and X_m by covariant derivatives of $U(N)$ etc. The theory can be written as $U(N)$ gauge theory on $\hat{T}^4/Z_2$.

$$S = \int_{\hat{T}^4/Z_2} \left\{ -\frac{1}{4g} \mathrm{tr} F_{ab} F^{ab} - \frac{1}{2g} D_a X^i D^a X_i \right\}. \tag{5.2}$$

There is no non-trivial gauge transformation accompanying reflection in the fixed points, so the fields are sections of a trivial bundle.

5.1. The interpretation on T^4/Z_2

This representation can be interpreted in terms of a two-brane wrapped at the origin in T^4/Z_2. This is inspired by the interpretation of a similar representation in the case of R^4/Z_2 in terms of a two-brane at the fixed point [24, 37]. More generally we have N two-branes at the origin. This is not the only possible interpretation but appears to be the simplest one consistent with the moduli spaces of vacua. Because of the B-field at the fixed point, the two-brane induces a zero-brane charge of 1/2 via the coupling (2.6).

An obvious check of this interpretation is the appearance of the expected $U(N)$ gauge symmetry of N identical objects. Further evidence for this picture will be given in section 6, where we describe representations of the algebra (4.3) which correspond to two-branes at other fixed points than the origin.

5.2. The moduli space

The moduli space has no Higgs branch. The constraint that A is odd rules out the constant connections which, in Rep. I, describe a T^4/Z_2. This is consistent with the picture of a two-brane stuck at the fixed point.

There is a Coulomb branch where X_i acquire constant expectation values, and we have a space R^5, or more generally $(R^5)^N/S_N$. This corresponds to the N two-branes moving in the transverse space.

5.3. Interpretation on $\hat{T}^4/Z_2$

We have described above the interpretation of the theory in terms of branes on T^4/Z_2. The form of the theory is also consistent with a description as a theory of N four-branes wrapped on $\hat{T}^4/Z_2$. Since the bundle is trivial, this is a quotient of N four-branes wrapped on $\hat{T}^4$ with the Z_2 action on space only, unlike Rep. I where the Z_2 is embedded non-trivially in the gauge group.

The moduli space is consistent with the picture that this representation describes a physical four-brane living on $\hat{T}^4/Z_2$, and Mukai charge $(N, 0, -N)$, with $B = -\frac{1}{4}\sum \gamma_i$ where γ_i are elements of $H^2(K3)$ corresponding to the 2-cycles collapsed at the fixed points. The Coulomb branch of $(R^5)^N/S_N$ is interpreted as the configuration space of positions of the four-branes in the transverse space.

6. General Reps — Branes at collapsed and smooth cycles

In general, we expect that there are representaions of the constraint algebra corresponding to different combinations of zero-branes, two-branes and four-branes on T^4/Z_2. In this section we will discuss the form of these generalized reps.

6.1. Branes at collapsed cycles — the interpretation

Since $\Omega^2 = 1$, a simple class of generalizations of Rep. I and Rep. II is given by choosing $\Omega = *\mathrm{diag}(1\cdots 1, -1\cdots -1)$, with k $(+1)$ eigenvalues, and l (-1) eigenvalues. The translations are still given by $U_a = e^{2\pi i R_a x^a}$. This is interpreted in terms of k states with charge $(1, -\frac{1}{2})$ and l states with charge $(-1, \frac{1}{2})$ at the fixed point at 0. Representations which correspond to two-branes wrapped at different fixed points are obtained by translating by half a lattice spacing in any direction. We note that $e^{-i\pi k_a x^a} * e^{i\pi k_a x^a} = *e^{2i\pi k_a x^a}$, so that reps corresponding to branes at different fixed points are simply obtained by replacing one or more 1's along the diagonal by $e^{2i\pi k_a x^a}$.

6.2. Branes at collapsed cycles — tests of interpretation

We will describe tests of the above interpretation using a few examples.

1) Consider first the theory of two membranes at a fixed point of T^4/Z_2. We expect $U(2)$ gauge symmetry for two identical branes, We expect no Higgs branch since the membranes are not free to move off the fixed point, and a Coulomb branch of $(R^5)^2/Z_2$. All these conditions are satisfied by the proposed theory which is just Rep. II with $N = 2$. This is a special case of the discussion in section 5.

2) A system of membrane and antimembrane at the origin has the same charges as a zero-brane, and was discussed as the $N = 1$ case in Rep. I. The pair can split off as a zero-brane which can wander on the T^4/Z_2, so there is a Higgs branch. When the zero-brane is at a generic point on the orbifold, we have a Coulomb branch of R^5. When it hits any of the fixed points, we get an enhanced Coulomb branch $(R^5)^2$.

3) Now consider two membranes and an antimembrane at the origin. This is described by choosing $\Omega = *\mathrm{diag}(1, 1, -1)$, which we will call Rep. III. Analyzing the moduli space, one finds that there is a Higgs branch of T^4/Z_2, corresponding to a membrane-antimembrane pair forming a zero-brane and moving off the fixed point. At the origin of the Higgs branch, there is a Coulomb branch of $(R^5)^2/Z_2 \times R^5$. At a generic point of the Higgs branch there is $(R^5)^2$. At a fixed point other than the origin, we expect $(R^5)^3$ (without the Z_2 quotient).

The origin of T^4/Z_2 corresponds to the trivial flat connection. We can pick a gauge where the points away from the origin correspond to connections of the form

$$\begin{pmatrix} 0 & 0 & B^a \\ 0 & 0 & 0 \\ B^a & 0 & 0 \end{pmatrix} .$$

This connection corresponding to the point at B^a on T^4/Z_2 is obtained by a "gauge-like" transformation from the trivial connection with gauge parameter, $U = e^{ix_a B^a (E_{13}+E_{31})}$, where $E_{13} + E_{31}$ is the matrix with 1 in the (13) slot and in the (31) slot. It is not a real gauge transformation on $\hat{T}^4$ because it is not periodic. If we act with this gauge transformation on a configuration of the transverse coordinates X^i, with non-zero components along the diagonal, the resulting configuration does not satisfy periodic boundary conditions for generic B^a. Only if $X^i_{33} = -X^i_{11}$, are the boundary conditions are satisfied. For $B^a = \pi k^a$, we do not need this condition for periodicity of the X^i, so the moduli space is enhanced to $(R^5)^3$. There is no Z_2 modding because conjugating the permutation which exchanges the first and second entry by $e^{ix_a B^a (E_{13}+E_{31})}$ does not give a valid gauge transformation for $B^a = \pi k^a$. For multiples of $2\pi R_a$ of course we do get valid gauge transformations, and we recover the same Coulomb branch as at the origin of the T^4/Z_2.

4) A final example is one where we have $\Omega = *\mathrm{diag}(1, e^{2\pi i R_1 x^1}, -e^{2\pi i R_1 x^1})$. We will call this Rep. IV. This describes a membrane at the origin and a membrane-antimembrane pair at the fixed point $(\pi R_1, 0, 0, 0)$. The membrane-antimembrane pair

should be able to split off as a zero-brane so we expect a Higgs branch of T^4/Z_2. When the zero brane is at $(\pi R_1, 0, 0, 0)$ (corresponding to the trivial connection) we have an enhanced Coulomb branch of $(R^5)^3$. At the non-zero connection corresponding to the zero-brane hitting the origin, we have an enhanced Coulomb branch of $(R^5)^2/Z_2 \times R^5$.

These features can be obtained essentially by conjugating the solutions from before. We can relate solutions of Rep. III to those of Rep. IV by observing that the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(1+e^{2\pi i R_1 x_1}) & \frac{1}{2}(1-e^{2\pi i R_1 x_1}) \\ 0 & \frac{1}{2}(1-e^{2\pi i R_1 x_1}) & \frac{1}{2}(1+e^{2\pi i R_1 x_1}) \end{pmatrix} \tag{6.1}$$

conjugates Rep. III into Rep. IV. So the trivial connection in Rep. III, with its associated Coulomb branch of $(R^5)^2/Z_2 \times R^5$, maps to a non-zero connection in Rep. IV. The connection corresponding to the fixed point $(\pi R_1, 0, 0, 0)$ maps to the trivial connection in Rep. IV, again giving the desired Coulomb branch.

6.3. Branes at uncollapsed cycles

There is a natural generalization of the representations considered above. We considered above a class of representations where $\Omega = *T$, with $T^2 = 1$, and the gauge fields were restricted to be periodic or anti-periodic. We now show that each such choice of Ω admits generalizations which are characterized by quantities $\epsilon^{abcd}\mathrm{tr}(X_a X_b X_c X_d)$ and $\mathrm{tr}[X_a, X_b]$. In the absence of B field on the cycles of the torus, these are the Chern characters which are integral.

We will illustrate by considering the case $\Omega = *T$, where T is a constant matrix squaring to 1. X_a are still covariant derivatives, now acting on the space of sections of a non-trivial $U(2N)$ bundle on $\hat{T}^4$ with transition functions constrained by the form of Ω.

To be more concrete we specify some of the constraints that the patching functions must satisfy. The $\hat{T}^4$ is being realized as a quotient of R^4 by 4 translations. There is also a Z_2 quotient to give $\hat{T}^4/Z_2$. We can give the bundle on the orbifold as a quotient of a bundle on R^4. The 4 translations are accompanied by gauge transformations $\Omega_a(x)$. The reflection in the origin is accompanied by a gauge transformation T. Reflection in other fixed points of the $\hat{T}^4/Z_2$ are accompanied by gauge transformations constructed from Ω_a and T. For example a reflection around the fixed point at $(\pi R_1, 0, 0, 0)$ is accompanied by $\Omega_1 T$. Patching functions and gauge transformations, viewed as sections on R^4 obey the conditions :

$$\begin{aligned} g(x + 2\pi R_a) &= \Omega_a g(x) \Omega_a^{-1} \\ g(-x) &= T g(x) T^{-1} . \end{aligned} \tag{6.2}$$

The operator Ω is a map between the fibre at x and the fibre at $-x$ satisfying $\Omega \Omega_a \Omega = \Omega_a^{-1}$. The Ω_a allow non-trivial Chern Classes of the $\hat{T}^4$ bundle. T encodes non-trivial bundle structure associated with the fixed points of $\hat{T}^4/Z_2$.

6.4. Traces and branes

We discuss in this section the relation between traces of operators appearing in (4.3) in these representations of the algebra (4.1) and brane charges on T^4/Z_2. For simplicity we restrict to the case where the B-field is zero on the cycles of the torus.

By analogy with the torus case, the operators $\epsilon^{abcd}\mathrm{tr}(X_aX_bX_cX_d)$, $\mathrm{tr}[X_a, X_b]$ and $\mathrm{tr}1$ measure four-brane charge, two-brane charge on the uncollapsed cycles of T^4/Z_2, and zero-brane charge. As the above examples suggest, the two-brane charges at the fixed points $\pi\epsilon_a R_a$ are measured by $\mathrm{tr}(T\prod_a U_a^{\epsilon_a})$ where ϵ_a is 0 or 1. There are some subtleties in the exact map between the traces and brane charges that remain to be worked out. We expect from the example of Rep. II (but have not given a general argument) that the zero-brane charge actually includes, in the presence of two-branes on the collapsed cycles, the induced charge due to the B-field according to (2.6). It remains to be determined whether, in the presence of four-branes, it includes the induced charge due to the curvature of the orbifold.

It is also natural to ask for the interpretation of the traces in terms of the brane charges on $\hat{T}^4/Z_2$. This would require a more systematic understanding of the relation between the theory on T^4/Z_2 and the one on $\hat{T}^4/Z_2$. We will turn to this in the next section.

7. Identifying the duality

We have seen in section 3 that T-duality of T^4 should lead to a map between the theory on an orbifold T^4/Z_2 of volume V and a theory on $\hat{T}^4/Z_2$ of volume $1/4V$. There is also the Fourier-Mukai (FM) transformation which exchanges four-brane and zero-brane Mukai charges and, at least without the presence of B-field, inverts the volume of the orbifold.

The above discussion of various representations of the algebra (4.3) always culminates in a theory of four-branes on $\hat{T}^4/Z_2$. It may also contain other brane charges. In particular, in Rep. I the appearance of a non-trivial gauge bundle due to the embedding of the Z_2 symmetry in the gauge group suggests there are membrane and possibly zero-brane charges at the fixed points. From the moduli space, we have argued that the different representations correspond to different collections of zero-branes on the original T^4/Z_2, together with membranes on the collapsed cycles at the fixed points. It is natural to ask if there is a K3 duality which realizes the transformation between a theory of zero-branes and membranes on T^4/Z_2 to one of four-branes and perhaps other brane charges on the dual $\hat{T}^4/Z_2$.

It is clear that the duality is not Fourier-Mukai. This is apparent from the fact that Rep. I which describes a single zero-brane on T^4/Z_2 is related to a $U(2)$ gauge theory on the dual space. This means that a zero-brane is dual to *two* four-branes, with possibly extra two-brane and zero-brane charges. Furthermore, we find that the volume of the dual orbifold is $\bar{V} = 1/4V$. Without B-field Fourier-Mukai duality sets

$\bar{V} = 1/V$. With B-field the situation is worse. As discussed in Appendix A, the volume is not in general inverted. Furthermore the B-field at the fixed point is rescaled, so the dual orbifold is not a perturbative string background. We will, however return to this duality in the next section, combining it with some extra assumptions to learn something about instantons on T^4.

The fact that the zero-brane maps to two four-branes strongly suggests that the duality in question is that inherited from the action of T-duality of T^4. As discussed in section 3 and Appendix A, for a system with Mukai charge (Q_4, Q_2, Q_0) we can associate a vector in the lattice $\Gamma^{4,20}$. Dualities are discrete rotations O which preserve the lattice. From Rep. I we have that one zero-brane maps to two four-branes and perhaps additional membrane and zero-brane charges at the fixed points. As discussed in Appendix A, requiring that the transformation inverts the volume and preserves the B-field flux at each fixed point implies that

$$\omega = C\left\{2\bar{\omega}^* + 2\bar{\omega} - \frac{1}{2}\left(\bar{\gamma}_1 + \cdots + \bar{\gamma}_{16}\right)\right\} \tag{7.1}$$

for some integer C, where $\bar{\gamma}_i$ are the sixteen collapsed two-cycles. Fractional amounts of $\bar{\gamma}_i$ are allowed because they span the Kummer lattice rather than $\Gamma^{4,20}$. From Rep. I we expect that $C = 1$. One can then check that this transformation preserves the mass of the zero-brane according to the mass formula in Appendix, with $C = 1$, and $V \to 1/4V$. This expression satisfies the desired condition $\omega \cdot \omega = 0$. Another promising aspect of this transformation is that it mixes zero-brane and collapsed membrane charge, which is suggested by the non-trivial bundle structure of Rep. I.

The amount of zero-brane charge appearing on the dual orbifold is interesting. With $C = 1$, a single zero-brane transforms into two four-branes and 2 units of zero-brane charge. However, this is Mukai charge (see section 3 and Appendix). Since the curvature of the orbifold induces -2 units, the actual number of zero-branes is 4. This corresponds to $1/4$ of a zero-brane charge at each fixed point. Fractionally charged instantons at ADE singularities have been discussed in [15, 14]. The $U(1)$ instanton constructed in [14] has membrane charge specified by $\int_{C_i} F/2\pi = 1$, or equivalently $Q_2 = -\frac{1}{2}\gamma_i$, and instanton number $-1/4$. Following the correspondence discussed in the Appendix, this is physical zero-brane charge $1/4$. This means that the dual of the zero-brane following from (7.1) is consistent with explicit constructions of instantons at the fixed points.

Let us now turn to Rep. II. We have argued that this corresponds to a collapsed membrane at the origin of T^4/Z_2. It gives a trivial bundle on $\hat{T}^4/Z_2$, naturally interpreted as single physical four-brane which, because of the curvature of the orbifold, has Mukai vector $(1, 0, -1)$. Thus, if γ_{16} is the collapsed two-cycle at the origin, we have

$$\gamma_{16} = \pm\left(\bar{\omega}^* - \bar{\omega}\right), \tag{7.2}$$

since these arguments cannot fix the overall sign. What about membranes at other collapsed cycles? We argued in section six, that a membrane at the fixed point πk^a

where the components of k^a are 0 or R_a, is described by choosing $\Omega = *e^{2i\pi k_a x^a}$. If a vector with components q^a taking values $1/2R_a$ or zero, gives the coordinates of the fixed points in $\hat{T}^4/Z_2$, this means there is a non-trivial bundle structure around q^a if $2k_aq^a$ is odd. One finds that this implies that half the fixed points of $\hat{T}^4/Z_2$ have trivial bundle structure and half have non-trivial structure. Thus we expect that a membrane at a collapsed cycle other than the origin is mapped to a single four-brane together with non-trivial membrane charge at eight of the collapsed cycles of $\hat{T}^4/Z_2$, and, perhaps, some zero-brane charge. Consider a membrane at a fixed point (labelled by an index i running from 1 to 15) with coordinates πk^a in T^4/Z_2, with k^a not all zero. The set of fixed points of $\hat{T}^4/Z_2$ with non-trivial bundle structure are characterized by

$$\bar{D}_i = \sum_j \xi_i{}^j \bar{\gamma}_j \,. \tag{7.3}$$

Here, if the jth fixed point in $\hat{T}^4/Z_2$ has coordinates q^a, then $\xi_i{}^j$ is zero or one depending on whether $2k_aq^a$ is even or odd. One then notes that in fact $\frac{1}{2}\bar{D}_i$ is an element of the $\Gamma^{4,20}$ lattice, since the $\bar{\gamma}_i$ span the Kummer lattice rather than $\Gamma^{4,20}$. This suggests that membranes at fixed points other than the origin transform as

$$\gamma_i = \pm\left(\bar{\omega}^* + \bar{\omega} - \frac{1}{2}\bar{D}_i\right), \tag{7.4}$$

which, as required is orthogonal to γ_{16} and ω, and squares to -2. As with the expression for ω, we see that each membrane $\frac{1}{2}\bar{\gamma}_i$ in $\hat{T}^4/Z_2$ leads to $1/4$ unit of physical zero-brane charge, so that the expression is consistent with the explicit constructions of instantons at the fixed points [15, 14].

Thus the structure of the gauge theory for zero-branes and membranes at different collapsed cycles provides much of the information necessary to find the relevant duality. One finds the full transformation is given by

$$\begin{aligned}
\omega &= 2\bar{\omega}^* + 2\bar{\omega} - \frac{1}{2}\left(\bar{\gamma}_1 + \cdots + \bar{\gamma}_{15} + \bar{\gamma}_{16}\right) \\
\omega^* &= 2\bar{\omega}^* + 2\bar{\omega} - \frac{1}{2}\left(\bar{\gamma}_1 + \cdots + \bar{\gamma}_{15} - \bar{\gamma}_{16}\right) \\
\gamma_i &= \bar{\omega}^* + \bar{\omega} - \frac{1}{2}\bar{D}_i \qquad\qquad i = 1, \ldots, 15 \\
\gamma_{16} &= \bar{\omega}^* - \bar{\omega}\,.
\end{aligned} \tag{7.5}$$

This indeed preserves the lattice and maps T^4/Z_2 to $\hat{T}^4/Z_2$ with volume $1/4V$ and preserves the B-field flux on the collapsed cycles. To show this it is useful to note that

$$\bar{D}_i \cdot \bar{D}_j = -8 - 8\delta_{ij} \qquad \bar{\gamma}_{16} \cdot \bar{D}_i = 0\,. \tag{7.6}$$

One notes that the transformation (7.5) singles out one of the collapsed cycles γ_{16}, corresponding to the origin. Clearly there are really sixteen such dualities, each singling out a different fixed point. Here one particular duality was relevant because we chose an origin in solving the constraints.

8. Fourier-Mukai duality on $K3$ and point-like instantons on T^4

While the FM duality does not seem to be the right duality for relating the theory on T^4/Z_2 and $\hat{T}^4/Z_2$, it does allow us, if we make some further assumptions, to learn something about the moduli space of Z_2 invariant instantons on T^4.

For k zero-branes we have a moduli space $(T^4/Z_2)^k/S_k$ which appears as a space of vacua in the theory constructed using Rep. I. This is dual, by FM duality, to k four-branes with k instantons which is the theory constructed by representing the Z_2 by 1 in the gauge group as in Rep. II. This duality will actually change the B-field at the fixed point. If we assume, however, that the topology of the moduli space is not changed by shifting the B-field or the volume from one generic non-zero value to another, we can predict some properties of moduli spaces of instantons in theories constructed along the lines of Rep. II. We are led to look for configurations of instanton number $2k$ on T^4, which are invariant under the Z_2 projection. A subset of such instantons will be point-like. Their moduli space should not be larger than $(T^4/Z_2)^k/S_k$. We can, in fact, construct the symmetric product space by looking at k pairs of point-like instantons placed in a Z_2 invariant configuration on T^4. It follows then, that for this choice of instanton number, there are no Z_2 symmetric fat instantons. It might appear that the moduli space is actually larger since we could choose different embeddings of the point-like instantons in the $U(k)$. However it is known that on R^4, the Donaldson-Uhlenbeck compactification [38] does not keep track of the group orientations of the point-like instantons. (This has an interpretation in the context of the $(0,2)$ theory [39].) It is reasonable to conjecture then that this is also true for T^4. If we did keep track of the embedding of the instanton, there would be a subgroup of the gauge group which commutes with it, which could be measured by a non-trivial Wilson loop in the space with the point removed. Since $\pi_1(S^3)$ is trivial, there is no possibility of such non-trivial Wilson loop [15].

To get precise agreement with the symmetric product spaces, we would need to prescribe more carefully what kind of point-like instantons are allowed at the fixed points. The prescription which would work is to keep only those which can be obtained by taking a limit where point-like instantons on T^4/Z_2 approach from a smooth point to the fixed point. However there may be more general point-like instantons at the fixed points, suggesting that the moduli space is not left invariant if we combine FM duality with the shift of B-field and volume.

Further relations between Z_2 invariant instantons on T^4 for different ranks and instanton numbers can be guessed by similar arguments. Here we have less powerful statements because we do not have very explicit information about the relevant moduli spaces. Nevertheless, we are led to suggest that the space of Z_2 invariant instantons for gauge group $U(N)$ with instanton number $2k$ on T^4 (i.e. k on T^4/Z_2) has the same moduli space as $U(k-N)$ with instanton number $2k$, where we have, without loss of generality taken $k > N$.

9. The six-dimensional perspective and the ADS connection

Under the headings Rep. I and Rep. II, we studied, in sections 4 and 5, two different sectors of the theory of zero-branes on T^4/Z_2. They can be described as two different quotients of the theory of N four-branes on $\hat{T}^4$. For Rep. II, the group we are quotienting by is a Z_2 which acts only as reflection $x^a \to -x^a$ on the 4 spatial coordinates of the 4-brane. For Rep. I we have a Z_2 quotient, where a spatial reflection is combined with a gauge transformation. The gauge transformation squares to 1 and has an equal number of $+$ and $-$ eigenvalues. In this section it will be convenient to choose the gauge transformation as the $2N \times 2N$ matrix $\sigma_1 \otimes 1_{N\times N}$. The spatial reflection is, therefore, accompanied by exchanging half the branes with the other half.

In this section we will consider the limit where $\hat{T}^4$ decompactifies, i.e. the limit where T^4 goes to zero volume. Then we are led to consider the quotient of $2N$ four-branes of type IIA on $R^4 \times R^6$, by symmetries which involve reflection in 4 spatial coordinates of R^4. From the point of view of M-theory this is a system of $2N$ 5-branes.

In sections 4 and 5 we have really described quotients of 4+1 dimensional maximally supersymmetric Yang-Mills theory, which is the low-energy description of the theory of four-branes. This $4+1$ Yang Mills by itself is ill-defined in the UV. One simple way to embed it in a well-defined theory without gravity is to recognize four-branes as 5-branes with one leg wrapped along the eleventh direction. The strong coupling limit then decompactifies the extra dimension, to give the $(0,2)$ superconformal theory in 6 dimensions [40, 41].

The simplest way to make sense of these quotiented theories at all energy scales is to view them as quotients of the $(0,2)$ theory. The superconformal fixed point theory should have global $U(N)$ symmetry currents with computable correlators, so there is a well-defined sense in which they have $U(N)$ symmetry. It therefore makes sense to say that we can quotient them in two different ways depending on how the Z_2 is embedded in the gauge group.

We can also define the six dimensional limit as the strong coupling limit of the quotiented four-brane theory. This theory is not expected to have the full 6-dimensional Lorentz symmetry $SO(5,1)$ which is broken by the Z_2 quotient. However it is expected to have 6-dimensional *scale symmetry* because it is being claimed to be the six-dimensional fixed point theory that the 5-dimensional theory flows to. A discussion of the relation between scale and conformal invariance appears in [42] but it is not easy to compare it with this situation directly since, first, $SO(5,1)$ is broken, and secondly, we do not have an explicit Lagrangian description of the general $(0,2)$ theory.

A related way to understand the six-dimensional limit is to recognize that the momentum dual to the extra coordinate is just instanton number, as exploited in the context of Matrix Theory in [43]. We can expect that quantum mechanics on the moduli spaces of appropriate Z_2 invariant instantons will provide the Matrix Model definitions of these theories along the lines of [44, 45].

Since the large N limit of the five-brane theory is conjectured to be related to M theory on $ADS_7 \times S^4$, it is illuminating to look for quotients of this theory which correspond to the quotients in Rep. I and Rep. II. Quotients of ADS backgrounds have been considered in [46] in order to obtain information about string vacua with reduced supersymmetry. There the quotient acts on the sphere. Both quotients of interest here act on ADS_7 and break the $SO(6,2)$ conformal symmetry to $SO(4) \times SO(2,2)$. However there are important differences. We recall that ADS is a hyperboloid in an 8-dimensional embedding space

$$X_0^2 - \sum_{i=1}^{6} X_i^2 + X_7^2 = 1 \tag{9.1}$$

The metric on this space can be written in horospheric coordinates, (see for example [47]) appropriate to the five-brane, as

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + dx \cdot dx) \tag{9.2}$$

The quotient related to Rep. II acts simply as reversal of the 4 coordinates parallel to the four-brane. This is a reflection of x^a on the five-brane for $i = 1 \cdots 4$, which implies $X^a \to -X^a$. The fixed surface in this case is thus defined by the conditions $X^a = 0$, which gives a hyperboloid, which is ADS_3. The hyperboloid intersects infinity on S^2.

We propose that the quotient related to Rep. I acts as this reflection together with a reflection in z. This is motivated by an interesting picture of 'where the brane lives' on ADS space. Gibbons [47] observes that the metric written in horospheric coordinates (9.2) suggests an interpretation of the ADS solution as infinitely many branes aligned along the z coordinate. As we discussed at the beginning of this section, Rep. I corresponds to a quotient where a spatial reflection is accompanied by exchanging half the branes with the other half. So the natural proposal is that, on the ADS side, the reflection in x_i is accompanied by a reflection in z. In the hyperboloid coordinates this is equivalent to $X^0, X^5, X^6, X^7 \to -X^0, -X^5, -X^6, -X^7$. There are no fixed surfaces in the case of this quotient.

We present some pieces of evidence supporting this identification. First, it leaves unbroken the subgroup of $SO(4) \times SO(2,2)$. In terms of the embedded hyperboloid, the $SO(4)$ rotates $X_1 \cdots X_4$ while the $SO(2,2)$ rotates the coordinates X_0, X_5, X_6, X_7. The $SO(2,2)$ contains scale invariance which we expect from the argument we gave above. We can also decompose the $SO(6,2)$ algebra in terms of the 6-dimensional Lorentz group $SO(5,1)$, with generators $M_{\mu\nu}$ (rotations and boosts), P_μ (momenta), K_μ (special conformal transformations), D (dilatation). Then the surviving generators are M_{ab} which generate $SO(4)$, and $K_0, K_5, P_0, P_5, M_{05}, D$. It is easy to check that from the $SO(6,2)$ relations that these two sets commute and that the second set is isomorphic to $SO(2,2)$, the conformal algebra in $1+1$ dimensions. There is still the $Spin(5)$ R-symmetry. So the full bosonic symmetry group is $SO(4) \times SO(2,2) \times Spin(5)$.

The $SO(6,2)$ spinors decompose into $SO(5,1)$ spinors as

$$\begin{pmatrix} Q \\ S^c \end{pmatrix},$$

where $S^c = C(S^\dagger\Gamma_0)^T$. Q and S have opposite $SO(5,1)$ chirality. The reflection of X^0, X^5, X^6, X^7 projects the spinors onto spinors of definite chirality under $\Gamma^0\Gamma^5\Gamma^6\Gamma^7$ where these are $SO(6,2)$ Γ matrices. Since the $SO(6,2)$ spinors have definite $SO(6,2)$ chirality, their chirality under the $SO(2,2)$ operator $\Gamma^0\Gamma^5\Gamma^6\Gamma^7$, is correlated with their chirality under $\Gamma^1\Gamma^2\Gamma^3\Gamma^4$. Thus, since the surviving supersymmetries have definite $SO(2,2)$ chirality, they have definite $SO(4)$ chirality. Consequently, the number of supercharges in the algebra is 16.

We also observe that restricting the Q and S supercharges to positive $SO(4)$ chirality is consistent with the the truncation of the bosonic generators to those given above. The anti-commutators between Q or between S in the six-dimensional superconformal algebra, are given for example in [48],

$$\begin{aligned}
\{Q^i_{\alpha'}, Q^j_{\beta'}\} &= -2(\gamma_\mu)_{\alpha'\beta'}\Omega^{ij}P^\mu \\
\{S^i_\alpha, S^j_\beta\} &= -2(\gamma_\mu)_{\alpha\beta}\Omega^{ij}K^\mu\,.
\end{aligned} \tag{9.3}$$

One finds that when the spinors are of the same $SO(4)$ chirality, P_0, P_5, K_0, K_5 can occur on the right hand side, but P_i, K_i cannot.

There are also non-vanishing commutators of the form

$$\{Q^i_{\alpha'}, S^j_\beta\} = -2c_{\alpha'\beta}\left(\Omega^{ij}D + 4U^{ij}\right) - 2(\gamma_{\mu\nu})_{\alpha'\beta}\Omega^{ij}M^{\mu\nu}\,. \tag{9.4}$$

The $SO(4)$ chirality conditions on Q and S, then imply that only M_{05} and M_{ab} appear. If we look at the bosonic subalgebra, we have a product of three simple factors $SO(4)\times SO(2,2)\times Spin(5)$. If we look at the full superalgebra, however, these factors are not decoupled. The algebra is not simple. We know it cannot be, because a superalgebra with such a product as bosonic part does not appear in Kac's classification of Lie superalgebras [49] (a recent paper containing a review of some of the relevant facts is [50]).

The $SU(2)_R$ factor of $SO(4)$ and a similar $Sl(2)_R$ factor of $SO(2,2)$ act trivially on the spinors since we have spinors of a definite chirality under these two groups. So we can rewrite the algebra in terms of a decoupled $SU(2)_R\times Sl(2)_R$ part, together with a superalgebra whose bosonic part is $SU(2)_L\times Sl(2)_L\times Spin(5)$. So this is an interesting scale invariant fixed point in 6 dimensions which contains a product made of a bosonic algebra $SU(2)_R\times Sl(2)_R$ and a superalgebra, with 16 supercharges and a bosonic part $SU(2)_L\times Sl(2)_L\times Spin(5)$ as symmetry group. It will be interesting to give a more complete description of such superalgebras, containing scale invariance, and their relevance to higher dimensional branes, e.g. of the kind discussed in [51].

The unbroken bosonic symmetry group is the same in Rep. I and Rep. II. Furthermore, the surviving supersymmetry in both cases are also the same. In Rep. I we saw that the reflection projects onto spinors of positive chirality under $\Gamma^0\Gamma^5\Gamma^6\Gamma^7$. Since they had positive $SO(6,2)$ chirality, this implied they had positive chirality under the $SO(4)$ operator $\Gamma^1\Gamma^2\Gamma^3\Gamma^4$. For Rep. II, the reflections projects onto spinors of positive chirality under $\Gamma^1\Gamma^2\Gamma^3\Gamma^4$. Again this implies the supercharges have positive $SO(4)$ and $SO(2,2)$ chirality, and we are left with the same supersymmetry algebra in each case. From the point of view of the 5-brane world-volume theory, we are looking at different ways of embedding the Z_2 in the gauge group, which should not affect the surviving spacetime symmetries. This is analogous to the fact that expanding around different Wilson line backgrounds does not change the SUSY algebra.

Note that preserving the $SO(4) \times SO(2,2)$ symmetry constrains the class of scale invariant fixed points. The B-field at the fixed point which is a parameter in the weak coupling limit of the D-brane theory is no longer a physical parameter in the strong coupling limit. In other words, it is a perturbation of the four-brane theory which becomes irrelevant in the strong coupling limit. The blow-up parameters will break the $SO(4)$ symmetry. Thus, the only freedom we have is to add membrane charge. The ADS picture suggests that the description in terms of supergravity will become harder, with a more complicated fixed surface.

In the above discussion of Rep. I, we combined the spatial reflection with a symmetry J which represented the a non-trivial gauge transformation. It is in the centre of $SO(6,2)$, and satisfies $J^2 = 1$, and can be understood geometrically by viewing the hyperboloid as a double cover. To obtain generators of Z_k satisfying $J^k = 1$, as may be appropriate for more general ALE spaces, we may, as in [47] consider multiple covers.

10. Summary and conclusions

We have studied a class of representations of a non-commutative algebra in relation to branes on T^4/Z_2. The regular representation (Rep. I) described zero-branes on the orbifold. We checked that it gave the correct moduli spaces of vacua for both the Higgs and Coulomb branches. We have also found a description of the regular representation in terms of a quotient of the theory of four-branes on the dual torus, $\hat{T}^4$, which corresponds to a non-trivial bundle on $\hat{T}^4/Z_2$ (with singularities at the fixed points).

We also studied a class of representations corresponding to the addition of two-branes at different fixed points of T^4/Z_2. As for the regular representation we checked that the Coulomb and Higgs branches are as expected, and gave a formulation in terms of bundles on $\hat{T}^4/Z_2$.

The discussion of bundles on $\hat{T}^4/Z_2$ involves two algebras. The first one is obvious, it is just the *commutative* algebra of functions on $\hat{T}^4/Z_2$, i.e. even functions on the covering $\hat{T}^4$. There is another *non-commutative* algebra, the group algebra of the quotienting group which defines T^4/Z_2, which played an important role in describing the physics of branes and bundles on $\hat{T}^4/Z_2$.

Whereas, on tori, either commutative or non-commutative algebras can lead, via their regular representation, to a Higgs branch moduli spaces of $S^N(T^4)$, (depending on whether we have turned on B-fields or not), our examples strongly suggest that only a non-commutative algebra leads to moduli spaces of the type $S^N(T^4/Z_2)$. A second important role for the non-commutative algebra is that it provides operators whose traces are related to brane charges at the fixed point, as touched upon at the end of section 6.

We found the relation between the duality of the brane description on T^4/Z_2 to the one on $\hat{T}^4/Z_2$ and conventional T-dualities. We wrote down the relevant element of $O(\Gamma_{4,20})$. Interestingly it mixes twisted sector states with untwisted sector ones. This should be understandable using boundary states[2] along the lines of [52]. We deduced some properties of Z_2 symmetric instantons on T^4 using K3 dualities.

We observed that quotients of four-brane theories are very useful in describing the physics of branes on the orbifold. Regarding these as 5-branes in M theory, we identified some of the relevant quotients on the ADS side. The identification of the ADS background related to Rep. II is straightforward, The identification of the background corresponding to Rep. I is less straightforward, because the spatial reflection is accompanied by an action on the gauge indices labeling the brane. We proposed an identification based on the picture [47] that the branes are stacked along the z-coordinate (9.2). According to this picture, the answer to the question of where in ADS space the branes live is, colloquially, 'everywhere'. We analyzed the surviving bosonic and fermionic symmetries of the quotient and argued that they are expected physically.

Note added: After this paper was submitted, a related paper [53] appeared, having some overlap with our discussion of the moduli space of zero-branes and including an interesting discussion of blow-ups of the orbifold.

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A. Review of and remarks on the dualities of K3 compactifications of type IIA strings

In this appendix we summarize the moduli space of type IIA strings on K3 following Aspinwall [13]. In addition, we derive a mass formula for the BPS brane configurations,

[2] We are grateful for an illuminating discussion with Ashoke Sen on this subject.

and discuss duality in the presence of B-field. Most of the discussion is for a general K3, but in the final part we specialize, deriving conditions on the form of a duality transformation which preserves the perturbative T^4/Z_2 orbifold, up to inverting the volume.

The total integer cohomology $H^*(K3,Z) = H^0(K3,Z) + H^2(K3,Z) + H^4(K3,Z)$ forms a lattice $\Gamma_{4,20}$. Elements of the lattice are associated to cycles as explained in [13]. The inner product $u \cdot v$ between two elements of the lattice comes from the oriented intersection number of the corresponding cycles,

$$u \cdot v = \int_{K3} u \wedge v\,. \tag{A.1}$$

Thus, for instance, if ω generates $H^0(K3,Z)$ and ω^* generates $H^4(K3,Z)$ we have

$$\omega \cdot \omega = \omega^* \cdot \omega^* = 0\,, \qquad \omega \cdot \omega^* = 1\,. \tag{A.2}$$

It is natural to decompose the lattice into the orthogonal sublattices $H^2(K3,Z)$ and $H^0(K3,Z) + H^4(K3,Z)$,

$$\Gamma_{4,20} = \Gamma_{3,19} + \Gamma_{1,1}. \tag{A.3}$$

The gravitational and B-field moduli of the string theory are fixed by giving a spatial four-plane Π in the total cohomology over the reals, $H^*(K3,R) \cong R^{4,20}$. Note that $\Gamma_{4,20} = H^*(K3,Z) \subset H^*(K3,R)$ is a lattice in $R^{4,20}$. The Einstein metric and B-field can be extracted from the plane in the following way.

First one splits $R^{4,20} = R^{3,19} + R^{1,1}$ into $H^2(K3,R)$ and $H^0(K3,R) + H^4(K3,R)$, as was done for the lattice in (A.3). Since $R^{3,19}$ has only three spacelike directions, the projection Σ of Π into $R^{3,19}$ must be a spatial three-plane. This plane fixes the Einstein metric on K3 up to the overall volume. It can be described in terms of three orthonormal elements of $H^2(K3,R)$, which we will call s_i with $i = 1,2,3$. These components represent the real and imaginary parts of the complex structure and the Kähler form scaled so that volume of the K3 is one. Making rotations between the s_i defines the same plane and this represents the freedom to rotate between the sphere of complex structures S^2 on the same hyper-Kähler manifold. To define the full four-plane we need to give an additional vector lying at least partly in the $R^{1,1}$ space. Aspinwall [13] introduces the vector

$$B' = \omega^* + \alpha\omega + B \tag{A.4}$$

normalized such that $\omega \cdot B' = 1$ and where $B \in H^2(K3,R) \cong R^{3,19}$. For B' to be spacelike,

$$B' \cdot B' \equiv 2V = 2\alpha + B \cdot B > 0\,, \tag{A.5}$$

where the positive quantity $B \cdot B + 2\alpha$ has been identified with the volume V of the $K3$ manifold, while $B \in H^2(K3,R)$ then gives the value of the B-field in string units.

Evidence for this identification will come from calculating the masses of various BPS brane states. The full four-plane Π is then described by the span of the set of orthogonal vectors

$$B', \qquad \hat{s}_i = s_i - (B' \cdot s_i)\omega\,. \tag{A.6}$$

The actual moduli space is a little smaller than simply choosing a four-plane in $R^{4,20}$. One must identify planes which are related by the discrete group of rotations which preserve the lattice $O(\Gamma_{4,20})$.

The lattice also provides a simple labeling of BPS states. From the heterotic–type IIA duality, on the heterotic side, a given point in the lattice is a supersymmetric state of fixed winding and momentum. Under duality, these should map to supersymmetric states in the IIA theory. Since on the IIA side the lattice points are elements of the integer cohomology, the corresponding BPS states are naturally combinations of four-branes, membranes and zero-branes wrapping the different cycles of the K3. In general we can write a point in the lattice as

$$p = Q_0\omega + Q_2 + Q_4\omega^*\,, \tag{A.7}$$

where $Q_2 \in H^2(K3, Z)$. We will find that Q_4 is equal to the number of four-branes in the BPS state and Q_2 gives the number of membranes wrapping the two-cycles of the K3. However Q_0 equals the number of zero-branes minus the number of four-branes. This shift is due to the zero-brane charge induced by the presence of four-branes due to the curvature of the K3. In other words, p is the Mukai charge.

From the heterotic point of view, the projection of p onto the plane Π gives the momentum of the state in the (conventionally right-moving) supersymmetric sector, while the component of p perpendicular to the plane gives the momentum in the bosonic sector. From this we see that the square of the projection of p onto Π should give the mass of the BPS state in the heterotic string frame. Thus, as a test of the interpretation of the K3 moduli space, we calculate this mass on the type IIA side. This can then be compared with the known expression for the total Ramond-Ramond (RR) charges of a collection of branes. Projecting onto Π and squaring we find

$$m^2 = \frac{1}{V}\left\{\left[Q_4 V + Q_0 + Q_2\cdot B - \frac{1}{2}Q_4 B\cdot B\right]^2 + V\sum_i \left[Q_2\cdot s_i - Q_4 B\cdot s_i\right]^2\right\}\,. \tag{A.8}$$

The above formula has been written with a special choice of units. To make everything explicit, we should understand

$$V = \frac{V_{K3}}{(4\pi^2\alpha')^2}\,, \tag{A.9}$$

where V_{K3} is the physical volume of the K3. Further, there is an overall factor of $l_{het}^{-2} = \frac{e^{-2\phi_6}}{\alpha' V}$. We also have the relation between the ten-dimensional dilaton ϕ and ϕ_6 as $e^{-2\phi_6} = Ve^{-2\phi}$.

We can then read off the amount of four-, two- and zero-brane charge from their respective contributions to the mass. We find

$$\begin{aligned} q_4 &= Q_4 \\ q_2 &= Q_2 - Q_4 B \\ q_0 &= Q_0 + Q_2 \cdot B - \frac{1}{2} Q_4 B \cdot B \,. \end{aligned} \tag{A.10}$$

This can be compared with the gauge theory description of the physical RR charges in the presence of B-field and on a curved manifold. With the normalization of B given above, we have, for a $U(N)$ bundle [34]

$$q = \mathrm{tr}\, e^{i(F/2\pi - B)} \sqrt{\hat{A}(K3)}\,, \tag{A.11}$$

so that

$$\begin{aligned} q_4 &= N \\ q_2 &= \frac{i}{2\pi} \mathrm{tr} F - iNB \\ q_0 &= -\frac{1}{8\pi^2} \mathrm{tr} F \wedge F - N + \frac{1}{2\pi} \mathrm{tr} F \wedge B - \frac{1}{2} NB \wedge B \,, \end{aligned} \tag{A.12}$$

where we have used $p_1 = 48$ for a K3. We see the dependence on the B field exactly matches (A.10), and $Q_4 = N$, $Q_2 = i\mathrm{tr}F/2\pi$, while $Q_0 = -\mathrm{tr}F \wedge F/8\pi^2 - N$. The shift in the last expression corresponds to the induced zero-brane charge due to the curvature of the K3. The inclusion of this term justifies the identification of the Q's with the Mukai charges. Note that the contribution to the RR charge due to the B-field, which is generally non-integer, is not included in the description of the state as a point in the lattice. The B-field contributions are due to the projection of p onto Π, and for the $\frac{1}{2} B \cdot B$ term, because we defined the volume as $\alpha + \frac{1}{2} B \cdot B$.

We now turn to the question of duality in the presence of B-field. We have seen in section 3 that Fourier-Mukai duality without B-field corresponds to the element of $O(\Gamma_{4,20})$ which exchanges of ω and ω^*, sending V to $1/V$. For simplicity, let us assume that $B \cdot s_i = 0$. Then the Fourier-Mukai transformation

$$\omega \to \bar{\omega} = \omega^* \qquad \omega^* \to \bar{\omega}^* = \omega \tag{A.13}$$

gives

$$B' = \bar{\omega} + \alpha \bar{\omega}^* + B \to \bar{B}' = \bar{\omega}^* + \bar{\omega}/\alpha + B/\alpha \,, \tag{A.14}$$

where we have rescaled so that $\bar{B}'$ has the correct normalization in the new basis. This means that

$$V \to \frac{V}{\left(V - \frac{1}{2} B \cdot B\right)^2}\,, \qquad B \to \frac{B}{V - \frac{1}{2} B \cdot B}\,, \tag{A.15}$$

so that in general the volume is not inverted, and B is rescaled.

Finally let us specialize to discuss duality transformations for the T^4/Z_2 orbifold limit of a K3. The orbifold has sixteen collapsed two-cycles γ_i, one at each fixed point, as well as six two-cycles inherited from the T^4. While these cycles are a basis for $H^2(K3, Q)$, they are not correctly normalized to generate $\Gamma_{3,19}$ [15]. Instead they generate a sublattice called the Kummer lattice. As discussed in section 3, to get a perturbative string background we must have B-flux on each of the sixteen collapsed two-cycles. In our normalization this gives:

$$B' = \omega^* + \alpha\omega - \frac{1}{4}\left(\gamma_1 + \cdots + \gamma_{16}\right) . \tag{A.16}$$

Since $\gamma_i \cdot \gamma_j = -2\delta_{ij}$, we have $B \cdot B = -2$. Thus we immediately see that the Fourier-Mukai duality transformation given in (A.14), does not invert the volume. Instead, from (A.15), we find $V \to V\left(V+1\right)^{-2}$ and the flux on each collapsed cycle becomes $\frac{1}{2}\left(V+1\right)^{-1}$. Since the coefficient of γ_i is no longer 1/4, this does not represent a perturbative string background.

It is natural to ask what kind of transformation could invert the volume and preserves the B-flux on each collapsed cycle. Consider a transformation which rotates among the ω^*, ω and γ_i vectors. We assume

$$B' \to \bar{B}' = \bar{\omega}^* + \bar{\alpha}\bar{\omega} - \frac{1}{4}\left(\bar{\gamma}_1 + \cdots + \bar{\gamma}_{16}\right) , \tag{A.17}$$

such that the volume is inverted

$$V = \alpha - 1 \to \bar{V} = \bar{\alpha} - 1 = A/V \tag{A.18}$$

with some A. That this is true for all α immediately gives the result

$$\omega = C\left\{2\bar{\omega}^* + 2\bar{\omega} - \frac{1}{2}\left(\bar{\gamma}_1 + \cdots + \bar{\gamma}_{16}\right)\right\} \tag{A.19}$$

with C undetermined. It might appear that C is required to be a multiple of two if the transformation is to be symmetry of the lattice. However, in fact, $\frac{1}{2}\left(\bar{\gamma}_1 + \cdots + \bar{\gamma}_{16}\right)$ is an element of $H^2(K3, Z)$, a result of the Kummer lattice being a sublattice of $\Gamma_{3,19}$. Thus C is only required to be integer.

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